

A revised version of this paper will appear soon in
Annales de la Faculté des Sciences de Toulouse.

The Curie-Weiss Model of SOC in Higher Dimension

Matthias Gorny

Université Paris-Sud and ENS Paris

Abstract

We build and study a multidimensional version of the Curie-Weiss model of self-organized criticality we have designed in [2]. For symmetric distributions satisfying some integrability condition, we prove that the sum S_n of the random vectors in the model has a typical critical asymptotic behaviour. The fluctuations are of order $n^{3/4}$ and the limiting law has a density proportional to the exponential of a fourth-degree polynomial.

Résumé

Nous construisons et étudions une version multi-dimensionnelle du modèle d'Ising Curie-Weiss de criticalité auto-organisée que nous avons introduit dans [2]. Pour des distributions vérifiant une certaine condition d'intégrabilité, nous montrons que la somme S_n des variables aléatoires du modèle a un comportement asymptotique critique typique. Les fluctuations sont d'ordre $n^{3/4}$ et la loi limite admet une densité proportionnelle à l'exponentielle d'un polynôme de degré quatre.

AMS 2010 subject classifications: 60F05 60K35

Keywords: Ising Curie-Weiss, SOC, Laplace's method

1 Introduction

In [2] and [5], we introduced a *Curie-Weiss model of self-organized criticality* (SOC): we transformed the distribution associated to the generalized Ising Curie-Weiss model by implementing an automatic control of the inverse temperature which forces the model to evolve towards a critical state. It is the model given by an infinite triangular array of real-valued random variables $(X_n^k)_{1 \leq k \leq n}$ such that, for all $n \geq 1$, (X_n^1, \dots, X_n^n) has the distribution

$$\frac{1}{Z_n} \exp\left(\frac{1}{2} \frac{(x_1 + \dots + x_n)^2}{x_1^2 + \dots + x_n^2}\right) \mathbf{1}_{\{x_1^2 + \dots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i),$$

where ρ is a probability measure on \mathbb{R} which is not the Dirac mass at 0, and where Z_n is the normalization constant. We extended the study of this model in [7], [8], [6] and [9]. For symmetric distributions satisfying some exponential moments condition, we proved that the sum S_n of the random variables behaves as in the typical critical generalized Ising Curie-Weiss model: the fluctuations are of order $n^{3/4}$ and the limiting law is $C \exp(-\lambda x^4) dx$ where C and λ are suitable positive constants. Moreover, by construction, the model does not depend on any external parameter. That is why we can conclude it exhibits the phenomenon of self-organized criticality (SOC). Our motivations for studying such a model are detailed in [2].

Let $d \geq 1$. In this paper we define a d -dimensional version of the Curie-Weiss model of SOC, i.e, such that the X_n^k , $1 \leq k \leq n$, are random vectors in \mathbb{R}^d . Let us start by defining the *d -dimensional generalized Ising Curie-Weiss model*. Let ρ be a symmetric probability measure on \mathbb{R}^d such that

$$\forall v \geq 0 \quad \int_{\mathbb{R}^d} \exp(v\|z\|^2) d\rho(z) < \infty.$$

Assume that its covariance matrix

$$\Sigma = \int_{\mathbb{R}^d} z^{\dagger} z d\rho(z)$$

is invertible. It is known to be equivalent to non-degeneracy of ρ , i.e. that there no hyperplane has full measure. The d -dimensional generalized Ising Curie-Weiss model associated to ρ and to the temperature field T (which is here a $d \times d$ symmetric positive definite matrix) is defined through an infinite triangular array of random vectors $(X_n^k)_{1 \leq k \leq n}$ such that, for all $n \geq 1$, (X_n^1, \dots, X_n^n) has the distribution

$$\frac{1}{Z_n(T)} \exp\left(\frac{1}{2n} \langle T^{-1}(x_1 + \dots + x_n), (x_1 + \dots + x_n) \rangle\right) \prod_{i=1}^n d\rho(x_i),$$

where $Z_n(T)$ is a normalization. When $d = 1$ and $\rho = (\delta_{-1} + \delta_1)/2$, we recover the classical Ising Curie-Weiss model. Let $S_n = X_n^1 + \dots + X_n^n$ for any $n \geq 1$. By extending the methods of Ellis and Newmann (see [4]) to the higher dimension, we obtain that, under some « sub-Gaussian » hypothesis on ρ , if $T - \Sigma$ is a symmetric positive definite matrix, then

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}_d(0, T(T - \Sigma)^{-1}\Sigma),$$

the centered d -dimensional Gaussian distribution with covariance matrix $T(T - \Sigma)^{-1}\Sigma$. If $T = \Sigma$ (critical case) then

$$\frac{S_n}{n^{3/4}} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} C_\rho \exp(-\phi_\rho(s_1, \dots, s_d)) ds_1 \cdots ds_d,$$

where C_ρ is a normalization constant and ϕ_ρ is an homogeneous polynomial of degree four in $\mathbb{R}[X_1, \dots, X_d]$ such that $\exp(-\phi_\rho)$ is integrable with respect to the Lebesgue measure on \mathbb{R}^d . Detailed proofs of these results are given in section 23 of [6]. These results highlight that the non-critical fluctuations are normal (in the Gaussian sense) while the critical fluctuations are of order $n^{3/4}$.

Now we try to modify this model in order to construct a d -dimensional SOC model. As in [2], we search an automatic control of the temperature field T , which would be a function of the random variables in the model, so that, when n goes to $+\infty$, T converges towards the critical value Σ of the model. We start with the following observation: if $(Y_n)_{n \geq 1}$ is a sequence of independent random vectors with identical distribution ρ , then, by the law of large numbers,

$$\frac{\widehat{\Sigma}_n}{n} \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \Sigma,$$

where

$$\forall n \geq 1 \quad \widehat{\Sigma}_n = X_n^1 t(X_n^1) + \cdots + X_n^n t(X_n^n).$$

This convergence provides us with an estimator of Σ . If we believe that a similar convergence holds in the d -dimensional generalized Ising Curie-Weiss model, then we are tempted to « replace T by $\widehat{\Sigma}_n/n$ » in the previous distribution. Hence, in this paper, we consider the following model:

The model. Let $(X_n^k)_{n \geq d, 1 \leq k \leq n}$ be an infinite triangular array of random vectors in \mathbb{R}^d such that, for any $n \geq d$, (X_n^1, \dots, X_n^n) has the distribution $\tilde{\mu}_{n,\rho}$, the probability measure on $(\mathbb{R}^d)^n$ with density

$$(x_1, \dots, x_n) \mapsto \frac{1}{Z_n} \exp \left(\frac{1}{2} \left\langle \left(\sum_{i=1}^n x_i t x_i \right)^{-1} \left(\sum_{i=1}^n x_i \right), \left(\sum_{i=1}^n x_i \right) \right\rangle \right)$$

with respect to $\rho^{\otimes n}$ on the set

$$D_n^+ = \left\{ (x_1, \dots, x_n) \in (\mathbb{R}^d)^n : \det \left(\sum_{i=1}^n x_i t x_i \right) > 0 \right\},$$

where

$$Z_n = \int_{D_n^+} \exp \left(\frac{1}{2} \left\langle \left(\sum_{i=1}^n x_i t x_i \right)^{-1} \left(\sum_{i=1}^n x_i \right), \left(\sum_{i=1}^n x_i \right) \right\rangle \right) \prod_{i=1}^n d\rho(x_i).$$

For any $n \geq d$, we denote $S_n = X_n^1 + \cdots + X_n^n \in \mathbb{R}^d$ and

$$T_n = X_n^1 t(X_n^1) + \cdots + X_n^n t(X_n^n).$$

According to the construction of this model and according to our results in one dimension, we expect that the fluctuations are of order $n^{3/4}$. Our main theorem states that they are indeed:

Theorem 1. Let ρ be a symmetric probability measure on \mathbb{R}^d satisfying the two following hypothesis:

(H1) there exists $v_0 > 0$ such that $\int_{\mathbb{R}^d} e^{v_0 \|z\|^2} d\rho(z) < \infty$,

(H2) the ρ -measure of any vector hyperplane of \mathbb{R}^d is less than $1/\sqrt{e}$.

Let Σ be the covariance matrix of ρ and let M_4 be the function defined on \mathbb{R}^d by

$$\forall z \in \mathbb{R}^d \quad M_4(z) = \int_{\mathbb{R}^d} \langle z, y \rangle^4 d\rho(y).$$

Law of large numbers: Under $\tilde{\mu}_{n,\rho}$, $(S_n/n, T_n/n)$ converges in probability to $(0, \Sigma)$.

Fluctuation result: Under $\tilde{\mu}_{n,\rho}$,

$$\frac{S_n}{n^{3/4}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \frac{\int_{\mathbb{R}^d} \exp\left(-\frac{1}{12} M_4(\Sigma^{-1}z)\right) dz}{\int_{\mathbb{R}^d} \exp\left(-\frac{1}{12} M_4(\Sigma^{-1}u)\right) du}.$$

We prove that the matrix Σ is invertible in subsection 2.a). In section 2.b), we prove rigorously that this model is well-defined, *i.e.* $Z_n \in]0, +\infty[$ for any $n \geq d$. After giving large deviation results in subsection 2.c), we show the law of large numbers in section 3. Finally, in section 4, we prove that the function

$$z \mapsto \exp\left(-M_4\left(\Sigma^{-1/2}z\right)/12\right)$$

is integrable on \mathbb{R}^d and that $S_n/n^{3/4}$ converges in distribution to the announced limiting distribution.

Remark : in the case where $d = 1$, we have already proved this theorem in [2], [8] and [9]. Moreover we succeeded to remove hypothesis (H2) – which turns out to be simply $\rho(\{0\}) < 1/\sqrt{e}$ when $d = 1$ – with a conditioning argument. It seems not immediate that such arguments could extend in the case where $d \geq 2$. However this assumption together with hypothesis (H1) are technical hypothesis and we believe that the result should be true if ρ is only a non-degenerate symmetric probability measure on \mathbb{R}^d having a finite fourth moment.

2 Preliminaries

In this section, we suppose that ρ is a symmetric probability measure on \mathbb{R}^d satisfying hypothesis (H1) and (H2).

a) Σ is a symmetric positive definite matrix

Since ρ satisfies hypothesis (H1), the covariance matrix Σ is well-defined. It is of course a symmetric positive semi-definite matrix. Let \mathcal{H} be a hyperplane of \mathbb{R}^d . If \mathcal{H} is a vector hyperplane then, by hypothesis, $\rho(\mathcal{H}) < 1/\sqrt{e} < 1$. If \mathcal{H} is an affine (but not vector) hyperplane then,

$$\rho(\mathcal{H}) = \rho(-\mathcal{H}) = \frac{1}{2}(\rho(\mathcal{H}) + \rho(-\mathcal{H})) \leq \frac{1}{2} < 1,$$

since ρ is symmetric and $\mathcal{H} \cap (-\mathcal{H}) = \emptyset$. In both cases $\rho(\mathcal{H}) < 1$ thus ρ is a non-degenerate probability measure on \mathbb{R}^d . As a consequence Σ is positive definite.

Notice that the hypothesis that $\rho(\mathcal{H}) < 1/\sqrt{e}$ is not involved on this point. We only need that ρ is non-degenerate.

b) The model is well-defined

Let us prove that the model is well defined, *i.e.* $Z_n \in]0, +\infty[$ for any $n \geq d$.

Lemma 2. *Let $n \geq 1$ and let x_1, \dots, x_n be vectors in \mathbb{R}^d . We denote*

$$A_n = x_1 {}^t x_1 + \dots + x_n {}^t x_n.$$

- ★ *If $n < d$, then A_n is non-invertible.*
- ★ *If $n = d$, then A_n is invertible if and only if (x_1, \dots, x_n) is a basis of \mathbb{R}^d .*
- ★ *If $n > d$ and if the vectors x_1, \dots, x_n span \mathbb{R}^d , then A_n is invertible.*

Proof. ★ Let $n \leq d$. If $n < d$, we put $x_{n+1} = \dots = x_d = 0$. We denote by B the $d \times d$ matrix such that its columns are x_1, \dots, x_d . We have then, for any $1 \leq k, l \leq d$,

$$(B {}^t B)_{k,l} = \sum_{i=1}^d B_{k,i} B_{l,i} = \sum_{i=1}^d x_i(k) x_i(l) = \sum_{i=1}^d (x_i {}^t x_i)_{k,l} = (A_n)_{k,l}.$$

Therefore $A_n = B {}^t B$ and thus A_n is invertible if and only if B is invertible. As a consequence A_n is invertible if and only if (x_1, \dots, x_d) is a basis of \mathbb{R}^d . In the case where $n < d$, B has at least a null column and thus is not invertible.

★ Let $n > d$ and assume that the vectors x_1, \dots, x_n span \mathbb{R}^d . Then there exists then $1 \leq i_1 < \dots < i_d \leq n$ such that $(x_{i_1}, \dots, x_{i_d})$ is a basis of \mathbb{R}^d . As a consequence, by the previous case, A_n is the sum of a symmetric positive definite matrix and $n - d$ other symmetric positive semi-definite matrices. Therefore A_n is definite thus invertible. \square

Let $n \geq d$. The non-degeneracy of ρ implies that its support is not included in a hyperplane of \mathbb{R}^d . As a consequence

$$\rho^{\otimes n}(\{(x_1, \dots, x_n) \in (\mathbb{R}^d)^n : (x_1, \dots, x_d) \text{ is a basis of } \mathbb{R}^d\}) > 0.$$

The previous lemma yields

$$\rho^{\otimes n}(\{(x_1, \dots, x_n) \in (\mathbb{R}^d)^n : x_1 {}^t x_1 + \dots + x_n {}^t x_n \text{ is invertible}\}) > 0,$$

i.e. $\rho^{\otimes n}(D_n^+) > 0$. Therefore $Z_n > 0$.

Let $\langle \cdot, \cdot \rangle$ be the usual scalar product on \mathbb{R}^d and $\|\cdot\|$ be the Euclidean norm. We denote:

- \mathcal{S}_d the space of $d \times d$ symmetric matrices.
- \mathcal{S}_d^+ the space of all matrices in \mathcal{S}_d which are positive semi-definite.
- \mathcal{S}_d^{++} the space of all matrices in \mathcal{S}_d which are positive definite.

We introduce the sets

$$\Delta = \{ (x, M) \in \mathbb{R}^d \times \mathcal{S}_d^+ : M - x {}^t x \in \mathcal{S}_d^+ \}.$$

and

$$\Delta^* = \{ (x, M) \in \mathbb{R}^d \times \mathcal{S}_d^{++} : M - x {}^t x \in \mathcal{S}_d^+ \}.$$

The two following lemmas guarantee that $Z_n < +\infty$ pour tout $n \geq 1$.

Lemma 3. *If $(x, M) \in \Delta^*$ then $\langle M^{-1}x, x \rangle \leq 1$.*

Proof. The matrix $M - x {}^t x$ is symmetric positive semi-definite. Hence

$$\forall y \in \mathbb{R}^d \quad \langle x, y \rangle^2 = \langle x {}^t x y, y \rangle \leq \langle M y, y \rangle.$$

Applying this inequality to $y = M^{-1}x$, we get

$$\langle x, M^{-1}x \rangle^2 \leq \langle M^{-1}x, x \rangle.$$

If $x = 0$ then $\langle M^{-1}x, x \rangle = 0 \leq 1$. If $x \neq 0$, since $M \in \mathcal{S}_d^{++}$, we have $\langle M^{-1}x, x \rangle > 0$ and thus $\langle M^{-1}x, x \rangle \leq 1$. \square

Let $n \geq 1$. For any $(x_1, \dots, x_n) \in (\mathbb{R}^d)^n$,

$$m = \frac{1}{n} \sum_{i=1}^n x_i \implies \frac{1}{n} \sum_{i=1}^n x_i {}^t x_i - m {}^t m = \frac{1}{n} \sum_{i=1}^n (x_i - m) {}^t (x_i - m) \in \mathcal{S}_d^+.$$

Therefore, for any $(x_1, \dots, x_n) \in D_n^+$,

$$\left(\frac{1}{n} \sum_{i=1}^n x_i, \frac{1}{n} \sum_{i=1}^n x_i {}^t x_i \right) \in \Delta^*$$

and thus

$$\frac{1}{2} \left\langle \left(\sum_{i=1}^n x_i {}^t x_i \right)^{-1} \left(\sum_{i=1}^n x_i \right), \left(\sum_{i=1}^n x_i \right) \right\rangle \leq \frac{n}{2}.$$

Hence $Z_n \leq e^{n/2} < +\infty$ and the model is well-defined for any $n \geq d$.

c) Large deviations for $(S_n/n, T_n/n)$

As in the one-dimensional case (see [2]), we introduce

$$F : (x, M) \in \Delta^* \longmapsto \frac{\langle M^{-1}x, x \rangle}{2}.$$

For any $n \geq d$, the distribution of $(S_n/n, T_n/n)$ under $\tilde{\mu}_{n,\rho}$ is

$$\frac{\exp(nF(x, M)) \mathbf{1}_{\{(x, M) \in \Delta^*\}} d\tilde{\nu}_{n,\rho}(x, M)}{\int_{\Delta^*} \exp(nF(s, N)) d\tilde{\nu}_{n,\rho}(s, N)},$$

where $\tilde{\nu}_{n,\rho}$ is the law of

$$\left(\frac{S_n}{n}, \frac{T_n}{n} \right) = \frac{1}{n} \sum_{i=1}^n (Y_i, Y_i {}^t Y_i)$$

when Y_1, \dots, Y_n are independent random vectors with common law ρ .

We endow $\mathbb{R}^d \times \mathcal{S}_d$ with the scalar product given by

$$((x, M), (y, N)) \mapsto \langle x, y \rangle + \text{tr}(MN) = \sum_{i=1}^d x_i y_i + \sum_{i=1}^d \sum_{j=1}^d m_{i,j} n_{i,j}.$$

We denote by $\|\cdot\|_d$ the associated norm. Notice that

$$\forall z \in \mathbb{R}^d \quad \forall A \in \mathcal{S}_d \quad \text{tr}(z {}^t z A) = \sum_{i=1}^d \sum_{j=1}^d z_i z_j a_{i,j} = \langle Az, z \rangle.$$

Let ν_ρ be the law of $(Z, Z {}^t Z)$ when Z is a random vector with distribution ρ . We define its Log-Laplace Λ , by

$$\begin{aligned} \forall (u, A) \in \mathbb{R}^d \times \mathcal{S}_d \quad \Lambda(u, A) &= \ln \int_{\mathbb{R}^d \times \mathcal{S}_d} \exp(\langle z, u \rangle + \text{tr}(MA)) d\nu_\rho(z, M) \\ &= \ln \int_{\mathbb{R}^d} \exp(\langle u, z \rangle + \langle Az, z \rangle) d\rho(z), \end{aligned}$$

and its Cramér transform I by

$$\forall (x, M) \in \mathbb{R}^d \times \mathcal{S}_d \quad I(x, M) = \sup_{(u, A) \in \mathbb{R}^d \times \mathcal{S}_d} (\langle x, u \rangle + \text{tr}(MA) - \Lambda(u, A)).$$

Let D_Λ and D_I be the domains of $\mathbb{R}^d \times \mathcal{S}_d$ where Λ and I are respectively finite. All these definitions generalize the case where $d = 1$, treated in [2] and [8].

For any $(u, A) \in \mathbb{R}^d \times \mathcal{S}_d$, we have

$$\begin{aligned} \exp \Lambda(u, A) &\leq \int_{\mathbb{R}^d} \exp(\|u\| \|z\| + \sqrt{\text{tr}(A^2)} \|z\|^2) d\rho(z) \\ &\leq \int_{\mathbb{R}^d} \exp(\|(u, A)\|_d \max(\|z\|, \|z\|^2)) d\rho(z) \\ &\leq \exp(\|(u, A)\|_d) + \int_{\mathbb{R}^d} \exp(\|(u, A)\|_d \|z\|^2) d\rho(z). \end{aligned}$$

Therefore hypothesis (H1) is sufficient to ensure that $(0, O_d)$ belongs to $\overset{\circ}{D}_\Lambda$, where O_d denotes the $d \times d$ matrix whose coefficients are all zero. As a consequence Cramér's theorem (cf. [3]) implies that $(\tilde{\nu}_{n,\rho})_{n \geq 1}$ satisfies the large deviation principle with speed n and governed by the good rate function I .

3 Convergence in probability of $(S_n/n, T_n/n)$

We saw in the previous section that, under the hypothesis of theorem 1, the sequence $(\tilde{\nu}_{n,\rho})_{n \geq 1}$ satisfies the large deviation principle with speed n and governed by the good rate function I . This result and Varadhan's lemma (see [3]) suggest that, asymptotically, $(S_n/n, T_n/n)$ concentrates on the minima of the function $I - F$. In subsection 3.a), we prove that $I - F$ has a unique minimum at $(0, \Sigma)$ on Δ^* and we extend F on the entire closed set Δ so that it remains true on Δ . This is the key ingredient for the proof of the law of large numbers in theorem 1, given in subsection 3.b).

a) Minimum de $I - F$

Proposition 4. *If ρ is a symmetric non-degenerate probability measure on \mathbb{R}^d , then*

$$\forall x \in \mathbb{R}^d \setminus \{0\} \quad \forall M \in \mathcal{S}_d^{++} \quad I(x, M) > \frac{\langle M^{-1}x, x \rangle}{2}.$$

Moreover, if Λ is finite in a neighbourhood of $(0, O_d)$, then the function $I - F$ has a unique minimum at $(0, \Sigma)$ on Δ^ .*

Proof. Let $x \in \mathbb{R}^d \setminus \{0\}$ and $M \in \mathcal{S}_d^{++}$. By taking $A = -M^{-1}x {}^t x M^{-1}/2$ and $u = M^{-1}x$, we get

$$\langle u, x \rangle + \text{tr}(AM) = \langle M^{-1}x, x \rangle - \frac{1}{2} \text{tr}(M^{-1}x {}^t x) = \frac{\langle M^{-1}x, x \rangle}{2}.$$

As a consequence

$$I(x, M) \geq \frac{\langle M^{-1}x, x \rangle}{2} - \Lambda \left(M^{-1}x, -\frac{1}{2}M^{-1}x {}^t x M^{-1} \right).$$

For any $z \in \mathbb{R}^d$, we have ${}^t z M^{-1}x = \langle M^{-1}x, z \rangle = \text{tr}(z {}^t (M^{-1}x)) \in \mathbb{R}$ thus

$$-\frac{1}{2} \text{tr}(z {}^t z M^{-1}x {}^t x M^{-1}) = -\frac{\langle M^{-1}x, z \rangle}{2} \text{tr}(z {}^t x M^{-1}) = -\frac{\langle M^{-1}x, z \rangle^2}{2}.$$

Therefore

$$\Lambda \left(M^{-1}x, -\frac{1}{2}M^{-1}x {}^t x M^{-1} \right) = \ln \int_{\mathbb{R}^d} \exp \left(\langle M^{-1}x, z \rangle - \frac{\langle M^{-1}x, z \rangle^2}{2} \right) d\rho(z).$$

By symmetry of ρ , we have, for any $s \in \mathbb{R}^d$,

$$\begin{aligned} \int_{\mathbb{R}^d} \exp \left(\langle s, z \rangle - \frac{\langle s, z \rangle^2}{2} \right) d\rho(z) &= \int_{\mathbb{R}^d} \exp \left(-\langle s, z \rangle - \frac{\langle s, z \rangle^2}{2} \right) d\rho(z) \\ &= \frac{1}{2} \left(\int_{\mathbb{R}^d} \exp \left(\langle s, z \rangle - \frac{\langle s, z \rangle^2}{2} \right) d\rho(z) + \int_{\mathbb{R}^d} \exp \left(-\langle s, z \rangle - \frac{\langle s, z \rangle^2}{2} \right) d\rho(z) \right) \\ &= \int_{\mathbb{R}^d} \cosh(\langle s, z \rangle) \exp \left(-\frac{\langle s, z \rangle^2}{2} \right) d\rho(z). \end{aligned}$$

As a consequence

$$\begin{aligned} \Lambda \left(M^{-1}x, -\frac{1}{2}M^{-1}x {}^t x M^{-1} \right) &= \\ &= \ln \int_{\mathbb{R}^d} \cosh(\langle M^{-1}x, z \rangle) \exp \left(-\frac{\langle M^{-1}x, z \rangle^2}{2} \right) d\rho(z). \end{aligned}$$

It is straightforward to see that the function $y \mapsto 1 - \cosh(y) \exp(-y^2/2)$ is non-negative on \mathbb{R} and vanishes only at 0. Hence, for any $z \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \cosh(\langle s, z \rangle) \exp \left(-\frac{\langle s, z \rangle^2}{2} \right) d\rho(z) \leq 1,$$

and equality holds if and only if $\rho(\{z : \langle s, z \rangle = 0\}) = 1$. The non-degeneracy of ρ implied that the equality case only holds if $s = 0$. Applying this to $s = M^{-1}x \neq 0$, we obtain

$$\Lambda \left(M^{-1}x, -\frac{1}{2}M^{-1}x {}^t x M^{-1} \right) < 0,$$

and thus $I(x, M) > \langle M^{-1}x, x \rangle / 2$.

Suppose now that $x = 0$ and $M \in \mathcal{S}_d^{++}$. Then

$$I(x, M) - \frac{\langle M^{-1}x, x \rangle}{2} = I(0, M).$$

If we assume that Λ is finite in a neighbourhood of $(0, \dots, 0, O_d)$, then $I(0, M) = 0$ if and only if $M = \Sigma$ (see proposition III.4 of [6]). This ends the proof of the proposition. \square

However, in order to apply Varadhan's lemma, F must be extended to an upper semi-continuous function on the entire closed set Δ . To this end, we put

$$\forall (x, M) \in \Delta \setminus \Delta^* \quad F(x, M) = \frac{1}{2},$$

and it is easy to check that F is indeed an upper semi-continuous function on Δ . Now we prove the inequality in proposition 4 holds on Δ .

Let $(x, M) \in \mathbb{R}^d \times \mathcal{S}_d^+$. We denote by $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_d$ the eigenvalues (not necessary distinct) of M . There exists an orthogonal matrix P such that $M = PD {}^t P$, where D is the diagonal matrix such that $D_{i,i} = \lambda_i$ for any $i \in \{1, \dots, d\}$. We have

$$\begin{aligned} I(x, M) &= \sup_{(u,A) \in \mathbb{R}^d \times \mathcal{S}_d} \left(\langle x, u \rangle + \text{tr}(PD {}^t P A) - \Lambda(u, A) \right) \\ &= \sup_{(u,A) \in \mathbb{R}^d \times \mathcal{S}_d} \left(\langle x, u \rangle + \text{tr}(DA) - \Lambda(u, PA {}^t P) \right). \end{aligned}$$

Assume that $M \notin \mathcal{S}_d^{++}$ and denote by $k = k_M \geq 1$ the dimension of the kernel of M . Let $a \in]-\infty, 0[$. By taking $u = 0$ and A the symmetric matrix such that

$$\forall (i, j) \in \{1, \dots, d\} \quad A_{i,j} = \begin{cases} a & \text{if } i = j \in \{1, \dots, k\}, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$I(x, M) \geq -\Lambda(u, PA {}^t P) = -\ln \int_{\mathbb{R}^d} \exp \langle PA {}^t P z, z \rangle d\rho(z),$$

i.e.

$$\forall a \in \mathbb{R} \quad I(x, M) \geq -\ln \int_{\mathbb{R}^d} \exp \left(a \sum_{j=1}^k ({}^t P z)_j^2 \right) d\rho(z).$$

For any $z \in \mathbb{R}^d$, we have

$$\sum_{j=1}^k ({}^t P z)_j^2 = 0 \iff z \in \text{Ker}(M)^\perp,$$

since (Pe_1, \dots, Pe_k) is a basis of $\text{Ker}(M)$ (they are the eigenvectors of M associated to the eigenvalue 0). As a consequence

$$\forall z \in \mathbb{R}^d \quad \exp \left(a \sum_{j=1}^k ({}^t Pz)_j^2 \right) \xrightarrow{a \rightarrow -\infty} \mathbb{1}_{\text{Ker}(M)^\perp}(z).$$

Moreover the left term defines a function which is bounded above by 1. Therefore the dominated convergence theorem implies that

$$\int_{\mathbb{R}^d} \exp \left(a \sum_{j=1}^k ({}^t Pz)_j^2 \right) d\rho(z) \xrightarrow{a \rightarrow -\infty} \rho(\text{Ker}(M)^\perp),$$

hence

$$I(x, M) \geq -\ln \rho(\text{Ker}(M)^\perp),$$

so that $I(x, M) > 1/2$ as soon as $\rho(\text{Ker}(M)^\perp) < e^{-1/2}$. Since $\text{Ker}(M)^\perp$ is included in some vector hyperplane of \mathbb{R}^d , we obtain the following proposition:

Proposition 5. *If ρ is a symmetric probability measure on \mathbb{R}^d satisfying hypothesis (H1) and (H2), then $I - F$ has a unique minimum at $(0, \Sigma)$ on Δ .*

b) Convergence of $(S_n/n, T_n/n)$ under $\tilde{\mu}_{n,\rho}$

Let us first prove the following proposition, which is a consequence of Varadhan's lemma.

Proposition 6. *Let ρ be a symmetric probability measure on \mathbb{R}^d with a positive definite covariance matrix Σ . We have*

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \ln Z_n \geq 0.$$

Suppose that ρ satisfies hypothesis (H1) and (H2). If \mathcal{A} is a closed subset of $\mathbb{R}^d \times \mathcal{S}_d$ which does not contain $(0, \Sigma)$, then

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^* \cap \mathcal{A}} \exp \left(\frac{n \langle M^{-1}x, x \rangle}{2} \right) d\tilde{\nu}_{n,\rho}(x, M) < 0.$$

Proof. The set $\overset{\circ}{\Delta^*}$, the interior of Δ^* , contains $(0, \Sigma)$ thus Cramér's theorem (cf. [3]) implies that

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln Z_n &= \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^*} \exp \left(\frac{n \langle M^{-1}x, x \rangle}{2} \right) d\tilde{\nu}_{n,\rho}(x, M) \\ &\geq \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \tilde{\nu}_{n,\rho}(\Delta^*) \geq -\inf \left\{ I(x, M) : (x, M) \in \overset{\circ}{\Delta^*} \right\} = 0. \end{aligned}$$

We prove now the second inequality. Since ρ verifies hypothesis (H1), we have that $(0, O_d) \in \overset{\circ}{D}_\Lambda$. Cramér's theorem implies then that $(\tilde{\nu}_{n,\rho})_{n \geq 1}$ satisfies the large deviation principle with speed n and the good rate function I . Since F is

upper semi-continuous on the closed set Δ , a variant of Varadhan's lemma (see Lemma 4.3.6 of [3]) yields

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^* \cap \mathcal{A}} \exp\left(\frac{n\langle M^{-1}x, x \rangle}{2}\right) d\tilde{\nu}_{n,\rho}(x, M) \\ \leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta \cap \mathcal{A}} \exp(nF(x, M)) d\tilde{\nu}_{n,\rho}(x, M) \leq \sup_{\Delta \cap \mathcal{A}} (F - I). \end{aligned}$$

Since ρ satisfies hypothesis (H2), proposition 5 implies that $I - F$ has a unique minimum at $(0, \Sigma)$ on Δ . Since the closed subset $\Delta \cap \mathcal{A}$ does not contain $(0, \Sigma)$ and since F is upper semi-continuous and I is a good rate function, we have

$$\sup_{\Delta \cap \mathcal{A}} (F - I) < 0.$$

This proves the second inequality of the proposition. \square

Proof of the law of large numbers in theorem 1. Suppose that ρ is symmetric and satisfies hypothesis (H1) and (H2). Let us denote by $\theta_{n,\rho}$ the law of $(S_n/n, T_n/n)$ under $\tilde{\mu}_{n,\rho}$. Let U be an open neighbourhood of $(0, \Sigma)$ in $\mathbb{R}^d \times \mathcal{S}_d$. Proposition 6 implies that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \theta_{n,\rho}(U^c) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^* \cap U^c} \exp\left(\frac{n\langle M^{-1}x, x \rangle}{2}\right) d\tilde{\nu}_{n,\rho}(x, M) \\ - \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln Z_n < 0. \end{aligned}$$

Hence there exist $\varepsilon > 0$ and $n_0 \geq 1$ such that $\theta_{n,\rho}(U^c) \leq e^{-n\varepsilon}$ for any $n \geq n_0$. Therefore, for any neighbourhood U of $(0, \Sigma)$,

$$\lim_{n \rightarrow +\infty} \tilde{\mu}_{n,\rho} \left(\left(\frac{S_n}{n}, \frac{T_n}{n} \right) \in U^c \right) = 0,$$

i.e. under $\tilde{\mu}_{n,\rho}$, $(S_n/n, T_n/n)$ converges in probability to $(0, \Sigma)$. \square

4 Convergence in distribution of $T_n^{-1/2} S_n/n^{1/4}$ under $\tilde{\mu}_{n,\rho}$

In this section, we generalize theorem 1 of [9] to the higher dimension in order to prove our fluctuation result.

Theorem 7. *Let ρ be a symmetric non-degenerate probability measure on \mathbb{R}^d such that*

$$\int_{\mathbb{R}^d} \|z\|^5 d\rho(z) < +\infty.$$

Let Σ the covariance matrix of ρ and let M_4 be the function defined in theorem 1. Then, under $\tilde{\mu}_{n,\rho}$,

$$\frac{1}{n^{1/4}} T_n^{-1/2} S_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \frac{\exp\left(-\frac{1}{12} M_4(\Sigma^{-1/2}z)\right) dz}{\int_{\mathbb{R}^d} \exp\left(-\frac{1}{12} M_4(\Sigma^{-1/2}u)\right) du}.$$

In the proof of this theorem, we show that the limiting law is well defined. Notice that, if $d = 1$, then $\Sigma^{-1/2} = \sigma^{-1}$ and

$$\forall z \in \mathbb{R} \quad M_4(\Sigma^{-1/2}z) = \frac{\mu_4 z^4}{\sigma^4}.$$

Hence theorem 7 is indeed a generalization of theorem 1 of [9]

a) Proof of theorem 7

Let $(X_n^k)_{n \geq d, 1 \leq k \leq n}$ be an infinite triangular array of random variables such that, for any $n \geq d$, (X_n^1, \dots, X_n^n) has the law $\tilde{\mu}_{n,\rho}$. Let us recall that

$$\forall n \geq 1 \quad S_n = X_n^1 + \dots + X_n^n \quad \text{and} \quad T_n = X_n^1 \text{ }^t(X_n^1) + \dots + X_n^n \text{ }^t(X_n^n).$$

and that $T_n \in \mathcal{S}_d^{++}$ almost surely. We use the Hubbard-Stratonovich transformation: let W be a random vector with standard multivariate Gaussian distribution and which is independent of $(X_n^k)_{n \geq d, 1 \leq k \leq n}$. Let $n \geq 1$ and f be a bounded continuous function on \mathbb{R}^d . We put

$$E_n = \mathbb{E} \left[f \left(\frac{W}{n^{1/4}} + \frac{1}{n^{1/4}} T_n^{-1/2} S_n \right) \right].$$

We introduce $(Y_i)_{i \geq 1}$ a sequence of independent random vectors with common distribution ρ . We denote

$$A_n = \sum_{i=1}^n Y_i, \quad B_n = \left(\sum_{i=1}^n Y_i \text{ }^t Y_i \right)^{1/2} \quad \text{and} \quad \mathcal{B}_n = \{ \det(B_n^2) > 0 \}.$$

We have

$$E_n = \frac{1}{Z_n (2\pi)^{d/2}} \mathbb{E} \left[\mathbf{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f \left(\frac{w}{n^{1/4}} + \frac{1}{n^{1/4}} B_n^{-1} A_n \right) \times \exp \left(\frac{1}{2} \left\langle B_n^{-2} A_n, A_n \right\rangle - \frac{\|w\|^2}{2} \right) dw \right].$$

We make the change of variables $z = n^{-1/4} (w + B_n^{-1} A_n)$ in the integral and we get

$$E_n = C_n \mathbb{E} \left[\mathbf{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp \left(-\frac{\sqrt{n} \|z\|^2}{2} + n^{1/4} \langle z, B_n^{-1} A_n \rangle \right) dz \right]$$

where $C_n = n^{d/4} Z_n^{-1} (2\pi)^{-d/2}$. Let $U_1, \dots, U_n, \varepsilon_1, \dots, \varepsilon_n$ be independent random variables such that the distribution of U_i is ρ and the distribution of ε_i is $(\delta_{-1} + \delta_1)/2$, for any $i \in \{1, \dots, n\}$. Since ρ is symmetric, the random variables $\varepsilon_1 U_1, \dots, \varepsilon_n U_n$ are also independent with common distribution ρ . Therefore

$$E_n = C_n \mathbb{E} \left[\mathbf{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp \left(-\frac{\sqrt{n} \|z\|^2}{2} + n^{1/4} \left\langle z, B_n^{-1} \left(\sum_{i=1}^n \varepsilon_i U_i \right) \right\rangle \right) dz \right].$$

In the case where the matrix $B_n^2 = U_1^t U_1 + \dots + U_n^t U_n$ is invertible, we denote

$$\forall i \in \{1, \dots, n\} \quad a_{i,n} = \left(\sum_{j=1}^n U_j^t U_j \right)^{-1/2} U_i.$$

By using Fubini's theorem and the independence of $\varepsilon_i, U_i, i \geq 1$, we obtain

$$E_n = C_n \mathbb{E} \left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp \left(-\frac{\sqrt{n} \|z\|^2}{2} \right) \times \mathbb{E} \left(\prod_{i=1}^n \exp \left(n^{1/4} \varepsilon_i \langle z, a_{i,n} \rangle \right) \middle| (U_1, \dots, U_n) \right) dz \right].$$

Therefore

$$E_n = C_n \mathbb{E} \left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp \left(-\frac{\sqrt{n} \|z\|^2}{2} \right) \exp \left(\sum_{i=1}^n \ln \cosh \left(n^{1/4} \langle z, a_{i,n} \rangle \right) \right) dz \right].$$

We define the function g by

$$\forall y \in \mathbb{R} \quad g(y) = \ln \cosh y - \frac{y^2}{2}.$$

It is easy to see that $g(y) < 0$ if $y > 0$. Therefore

$$\sum_{i=1}^n \langle z, a_{i,n} \rangle^2 = \sum_{i=1}^n \langle z, (a_{i,n}^t a_{i,n}) z \rangle = \left\langle z, \left(\sum_{i=1}^n a_{i,n}^t a_{i,n} \right) z \right\rangle = \langle z, \mathbf{I}_d z \rangle = \|z\|^2.$$

As a consequence

$$E_n = C_n \mathbb{E} \left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp \left(\sum_{i=1}^n g \left(n^{1/4} \langle z, a_{i,n} \rangle \right) \right) dz \right].$$

Now we use Laplace's method. Let us examine the convergence of the term in the exponential: for any $z \in \mathbb{R}^d$ and $i \in \{1, \dots, n\}$, the Taylor-Lagrange formula states that there exists a random variable $\xi_{n,i}$ such that

$$g \left(n^{1/4} \langle z, a_{i,n} \rangle \right) = -\frac{n \langle z, a_{i,n} \rangle^4}{12} + \frac{n^{3/2} \langle z, a_{i,n} \rangle^5}{n^{1/4} 5!} g^{(5)}(\xi_{n,i}).$$

Let $z \in \mathbb{R}^d$. We have

$$n \sum_{i=1}^n \langle z, a_{i,n} \rangle^4 = n \sum_{i=1}^n \langle B_n^{-1} z, U_i \rangle^4 = \frac{1}{n} \sum_{i=1}^n \langle \sqrt{n} B_n^{-1} z, U_i \rangle^4.$$

We denote $\zeta_n = \sqrt{n} B_n^{-1} z$. We have

$$\begin{aligned} n \sum_{i=1}^n \langle z, a_{i,n} \rangle^4 &= \frac{1}{n} \sum_{i=1}^n \langle \zeta_n, U_i \rangle^4 = \frac{1}{n} \sum_{i=1}^n \left(\sum_{j=1}^d (\zeta_n)_j (U_i)_j \right)^4 \\ &= \sum_{1 \leq j_1, j_2, j_3, j_4 \leq d} (\zeta_n)_{j_1} (\zeta_n)_{j_2} (\zeta_n)_{j_3} (\zeta_n)_{j_4} \frac{1}{n} \sum_{i=1}^n (U_i)_{j_1} (U_i)_{j_2} (U_i)_{j_3} (U_i)_{j_4}. \end{aligned}$$

Since ρ is non-degenerate, its covariance matrix Σ is invertible. Moreover ρ has a finite fourth moment thus the law of large numbers implies that

$$\zeta_n \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \Sigma^{-1/2} z,$$

and that, for any $(j_1, j_2, j_3, j_4) \in \{1, \dots, d\}^4$,

$$\frac{1}{n} \sum_{i=1}^n (U_i)_{j_1} (U_i)_{j_2} (U_i)_{j_3} (U_i)_{j_4} \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \int_{\mathbb{R}^d} y_{j_1} y_{j_2} y_{j_3} y_{j_4} d\rho(y).$$

As a consequence

$$n \sum_{i=1}^n \langle z, a_{i,n} \rangle^4 \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} M_4 \left(\Sigma^{-1/2} z \right).$$

Since ρ has a finite fifth moment, we prove similarly that

$$n^{3/2} \sum_{i=1}^n \langle z, a_{i,n} \rangle^5 \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} M_5 \left(\Sigma^{-1/2} z \right),$$

where $M_5(z) = \int_{\mathbb{R}^d} \langle z, y \rangle^5 d\rho(y)$ for any $z \in \mathbb{R}^d$. Finally, by a simple computation, we see that $g^{(5)}$ is bounded over \mathbb{R} . Hence

$$\forall z \in \mathbb{R}^d \quad \sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} -\frac{1}{12} M_4 \left(\Sigma^{-1/2} z \right).$$

Lemma 8. *There exists $c > 0$ such that*

$$\forall z \in \mathbb{R}^d \quad \forall n \geq 1 \quad \sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \leq -\frac{c \|z\|^4}{1 + \|z\|^2 / \sqrt{n}}.$$

Proof. We define h by

$$\forall y \in \mathbb{R} \setminus \{0\} \quad h(y) = \frac{1 + y^2}{y^4} g(y).$$

It is a non-negative continuous function on $\mathbb{R} \setminus \{0\}$. Since $g(y) \sim -y^4/12$ in the neighbourhood of 0, the function h can be extended to a function continuous on \mathbb{R} by putting $h(0) = -1/12$. Next we have

$$\forall y \in \mathbb{R} \setminus \{0\} \quad h(y) = \frac{1 + y^2}{y^2} \times \left(\frac{\ln \cosh y}{y^2} - \frac{1}{2} \right),$$

so that $h(y)$ goes to $-1/2$ when $|y|$ goes to $+\infty$. Therefore h is bounded by some constant $-c$ with $c > 0$. Hence, for any $z \in \mathbb{R}$ and $n \geq 1$,

$$\sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \leq -nc \frac{1}{n} \sum_{i=1}^n \frac{(n^{1/4} \langle z, a_{i,n} \rangle)^4}{1 + (n^{1/4} \langle z, a_{i,n} \rangle)^2}.$$

We easily check that $x \mapsto x^2/(1+x)$ is convex on $[0, +\infty[$. As a consequence

$$\sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \leq -nc \frac{\left(\frac{1}{n} \sum_{i=1}^n (n^{1/4} \langle z, a_{i,n} \rangle)^2 \right)^2}{1 + \frac{1}{n} \sum_{i=1}^n (n^{1/4} \langle z, a_{i,n} \rangle)^2} = -\frac{c \|z\|^4}{1 + \|z\|^2 / \sqrt{n}},$$

since $\langle z, a_{1,n} \rangle^2 + \dots + \langle z, a_{n,n} \rangle^2 = 1$. □

If $\|z\| \leq n^{1/4}$ then $1 + \|z\|^2/\sqrt{n} \leq 2$ and thus, by the previous lemma,

$$\left| \mathbb{1}_{\mathcal{B}_n} \mathbb{1}_{\|z\| \leq n^{1/4}} \exp \left(\sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \right) \right| \leq \exp \left(-\frac{c\|z\|^4}{2} \right).$$

Thus the dominated convergence theorem implies that

$$z \mapsto \exp \left(-M_4 \left(\Sigma^{-1/2} z \right) / 12 \right)$$

is integrable on \mathbb{R}^d and that

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} \mathbb{1}_{|z| \leq n^{1/4}} f(z) \exp \left(\sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \right) dz \right] \\ \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^d} f(z) \exp \left(-\frac{1}{12} M_4 \left(\Sigma^{-1/2} z \right) \right) dz. \end{aligned}$$

If $\|z\| > n^{1/4}$ then $1 + \|z\|^2/\sqrt{n} \leq 2\|z\|^2/\sqrt{n}$ and thus, by the previous lemma,

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} \mathbb{1}_{|z| > n^{1/4}} f(z) \exp \left(\sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \right) dz \right] \\ \leq \|f\|_\infty \int_{\mathbb{R}^d} \exp \left(-\frac{c\sqrt{n}\|z\|^2}{2} \right) dz = \frac{\|f\|_\infty (2\pi)^{d/2}}{n^{d/4} c^{d/2}} \xrightarrow{n \rightarrow +\infty} 0, \end{aligned}$$

and thus

$$\begin{aligned} \frac{E_n}{C_n} = \mathbb{E} \left[\mathbb{1}_{\mathcal{B}_n} \int_{\mathbb{R}^d} f(z) \exp \left(\sum_{i=1}^n g(n^{1/4} \langle z, a_{i,n} \rangle) \right) dz \right] \\ \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^d} f(z) \exp \left(-\frac{1}{12} M_4 \left(\Sigma^{-1/2} z \right) \right) dz. \end{aligned}$$

If we take $f = 1$, we get

$$\frac{1}{C_n} = \frac{Z_n (2\pi)^{d/2}}{n^{d/4}} \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^d} \exp \left(-\frac{1}{12} M_4 \left(\Sigma^{-1/2} z \right) \right) dz.$$

Summarizing, we have proved that

$$\frac{W}{n^{1/4}} + \frac{1}{n^{1/4}} T_n^{-1/2} S_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \frac{\exp \left(-\frac{1}{12} M_4 \left(\Sigma^{-1/2} z \right) \right) dz}{\int_{\mathbb{R}^d} \exp \left(-\frac{1}{12} M_4 \left(\Sigma^{-1/2} u \right) \right) du}.$$

Since $(Wn^{-1/4})_{n \geq 1}$ converges in distribution to 0, Slutsky's lemma (theorem 3.9 of [1]) implies the convergence in distribution of theorem 7.

We remark that we needed the hypothesis that ρ has a finite fifth moment in order to use Taylor-Lagrange formula. This hypothesis may certainly be weakened by assuming instead that

$$\exists \varepsilon > 0 \quad \int_{\mathbb{R}^d} \|z\|^{4+\varepsilon} d\rho(z) < +\infty.$$

b) Proof of the fluctuation result in theorem 1.

In section 3, we proved the law of large numbers in theorem 1. It implies that, under $\tilde{\mu}_{n,\rho}$, T_n/n converges in probability to Σ . Moreover hypothesis (H1) implies that $(0, O_d) \in \overset{\circ}{D}_\Lambda$ and thus ρ has finite moments of all orders. Theorem 7 and Slutsky lemma yield

$$\frac{S_n}{n^{3/4}} = \left(\frac{T_n}{n}\right)^{1/2} \times \frac{1}{n^{1/4}} T_n^{-1/2} S_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \frac{\int_{\mathbb{R}^d} \exp\left(-\frac{1}{12} M_4(\Sigma^{-1}z)\right) dz}{\int_{\mathbb{R}^d} \exp\left(-\frac{1}{12} M_4(\Sigma^{-1}u)\right) du}.$$

References

- [1] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., second edition, 1999. A Wiley-Interscience Publication.
- [2] Raphaël Cerf and Matthias Gorny. A Curie–Weiss model of self-organized criticality. *Ann. Probab.*, 44(1):444–478, 2016.
- [3] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, 2010.
- [4] Richard S. Ellis and Charles M. Newman. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Z. Wahrsch. Verw. Gebiete*, 44(2):117–139, 1978.
- [5] Matthias Gorny. A Curie-Weiss model of self-organized criticality: the Gaussian case. *Markov Process. Related Fields*, 20(3):563–576, 2014.
- [6] Matthias Gorny. *Un modèle d’Ising Curie–Weiss de criticité auto-organisée*. Phd thesis, Université Paris-Sud, 2015. <https://tel.archives-ouvertes.fr/tel-01167487>.
- [7] Matthias Gorny. A Dynamical Curie–Weiss model of SOC: the Gaussian case. *Ann. Inst. H. Poin.*, to appear, 2017.
- [8] Matthias Gorny. The Cramér condition for the Curie-Weiss model of SOC. *Braz. J. Probab. Stat.*, 30(3):401–431, 2016.
- [9] Matthias Gorny and S. R. S. Varadhan. Fluctuations of the self-normalized sum in the Curie-Weiss model of SOC. *J. Stat. Phys.*, 160(3):513–518, 2015.