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# Fluctuations of the Self-Normalized Sum in the Curie-Weiss Model of SOC

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**Abstract**

We extend the main theorem of [2] about the fluctuations in the Curie-Weiss model of SOC in the symmetric case. We present a short proof using the Hubbard-Stratonovich transformation with the self-normalized sum of the random variables.

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## 1 Introduction

In [2], Raphaël Cerf and Matthias Gorny designed a Curie-Weiss model of self-organized criticality. It is the model given by an infinite triangular array of real-valued random variables $(X_n^k)_{1\leq k\leq n}$ such that for all $n\geq 1$, $(X_n^1,\dots,X_n^n)$ has the distribution

$$d\widetilde{\mu}_{n,\rho}(x_1,\dots,x_n)=\frac{1}{Z_n}\exp\left(\frac{1}{2}\frac{(x_1+\dots+x_n)^2}{x_1^2+\dots+x_n^2}\right)\mathbb{1}_{\{x_1^2+\dots+x_n^2>0\}}\prod_{i=1}^n d\rho(x_i),$$

where $\rho$ is a probability measure on $\mathbb{R}$ which is not the Dirac mass at 0, and where $Z_n$ is the normalization constant. This model is a modification of the generalized Ising Curie-Weiss model by the implementation of an automatic control of the inverse temperature.

For any $n \geq 1$, we denote

$$S_n = X_n^1 + \cdots + X_n^n, \qquad T_n = (X_n^1)^2 + \cdots + (X_n^n)^2.$$

By using Cramér's theory and Laplace's method, Cerf and Gorny proved in [2] that, if $\rho$ satisfies

$$\exists v_0 > 0 \qquad \int_{\mathbb{R}} e^{v_0 z^2} \, d\rho(z) < +\infty \tag{$*$}$$

and if $\rho$ has a bounded density, then

$$\frac{S_n}{n^{3/4}} \xrightarrow[n\to\infty]{\mathscr{L}} \left(\frac{4\mu_4}{3\sigma^8}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{\mu_4}{12\sigma^8} s^4\right) ds.$$

The case where $\rho$ is a centered Gaussian measure has been studied in [7]. This fluctuation result shows that this model is a self-organized model exhibiting critical behaviour. Indeed it has the same behaviour as the critical generalized Ising Curie-Weiss model (see [5]) and, by construction, it does not depend on any external parameter.

This result has been extended in [6] to the case where $\rho$ satisfies some Cramér condition, which is fulfilled in particular when $\rho$ has a an absolutely continuous component. However the proof is very technical and it does not deal with the case where $\rho$ is discrete for example.

In this paper we prove that the convergence in distribution of $S_n/n^{3/4}$, under $\widetilde{\mu}_{n,\rho}$, is true for any symmetric probability measure $\rho$ on $\mathbb{R}$ which satisfies $(*)$. To this end, we study the fluctuations of the self-normalized sum $S_n/\sqrt{T_n}$. With this term, it is possible to use the so-called Hubbard-Stratonovich transformation as in lemma 3.3 of [5], which is the key ingredient for the proof of the fluctuations theorem in the generalized Ising Curie-Weiss model.

**Theorem 1.** *Let $\rho$ be a symmetric probability measure on $\mathbb{R}$ which is not the Dirac mass at $0$ and which has a finite fifth moment. We denote by $\sigma^2$ the variance of $\rho$ and by $\mu_4$ its fourth moment. Then, under $\widetilde{\mu}_{n,\rho}$,*

$$\frac{S_n}{n^{1/4}\sqrt{T_n}} \xrightarrow[n\to\infty]{\mathscr{L}} \left(\frac{4\mu_4}{3\sigma^4}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{\mu_4}{12\sigma^4} s^4\right) ds.$$

Remark: the hypothesis that $\rho$ has a fifth moment may certainly be weakened by assuming instead that

$$\exists \varepsilon > 0 \qquad \int_{\mathbb{R}} |z|^{4+\varepsilon} \, d\rho(z) < +\infty.$$

There is a huge literature on self-normalized sums that gives very precise results on ratios like $S_n/\sqrt{T_n}$, in the independent case (see for instance [3]). We prove theorem 1 in section 2, without requiring any preliminary result.

If we add the hypothesis that $\rho$ satisfies $(*)$ then, under $\widetilde{\mu}_{n,\rho}$, $T_n/n$ converges in probability to $\sigma^2$. This result is proved in section 3 of [6] using Cramér's theorem, Varadhan's lemma (see [4]) and a conditioning argument. Moreover

$$\forall n \geq 1 \qquad \frac{S_n}{n^{3/4}} = \sqrt{\frac{T_n}{n}} \times \frac{S_n}{n^{1/4}\sqrt{T_n}},$$

and condition $(*)$ implies that $\rho$ has finite moments of all orders. Therefore the following theorem is a consequence of theorem 1 and Slutsky lemma (theorem 3.9 of [1]).

**Theorem 2.** *Let $\rho$ be a symmetric probability measure on $\mathbb{R}$ which is not the Dirac mass at $0$ and such that*

$$\exists v_0 > 0 \qquad \int_{\mathbb{R}} e^{v_0 z^2}\, d\rho(z) < +\infty.$$

*Then, under $\widetilde{\mu}_{n,\rho}$,*

$$\frac{S_n}{n^{3/4}} \xrightarrow[n\to\infty]{\mathscr{L}} \left(\frac{4\mu_4}{3\sigma^8}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{\mu_4}{12\sigma^8}s^4\right)\, ds.$$

Remark: The assumption on the exponential moment of $\rho$ in the previous theorem is a technical hypothesis and we believe that the result should be true if $\rho \neq \delta_0$ is symmetric and has only a finite fourth moment. Nevertheless the symmetry of $\rho$ is essential even if $\rho$ has zero mean (see section 11.f. of [8] for the simple counter-example of a non-symmetric Bernoulli).

# 2 Proof of theorem 1

Let $(X_n^k)_{1\leq k\leq n}$ be an infinite triangular array of random variables such that, for any $n \geq 1$, $(X_n^1, \dots, X_n^n)$ has the law $\widetilde{\mu}_{n,\rho}$. Let us recall that

$$\forall n \geq 1 \qquad S_n = X_n^1 + \cdots + X_n^n \qquad \text{and} \qquad T_n = (X_n^1)^2 + \cdots + (X_n^n)^2,$$

and that $T_n > 0$ almost surely. We use the Hubbard-Stratonovich transformation: let $W$ be a random variable with standard normal distribution and which is independent of $(X_n^k)_{1\leq k\leq n}$. Let $n \geq 1$ and let $f$ be a bounded continuous function on $\mathbb{R}$. We put

$$E_n = \mathbb{E}\left[f\left(\frac{W}{n^{1/4}} + \frac{S_n}{n^{1/4}\sqrt{T_n}}\right)\right].$$

We introduce $(Y_i)_{i\geq 1}$ a sequence of independent random variables with common distribution $\rho$. We denote $A_n = Y_1 + \cdots + Y_n$ and $B_n = \sqrt{Y_1^2 + \cdots + Y_n^2}$. We have

$$E_n = \frac{1}{Z_n\sqrt{2\pi}}\,\mathbb{E}\left[\int_{\mathbb{R}} f\left(\frac{w}{n^{1/4}} + \frac{A_n}{n^{1/4}B_n}\right) \exp\left(\frac{1}{2}\frac{A_n^2}{B_n^2} - \frac{w^2}{2}\right) \mathbb{1}_{\{B_n^2>0\}}\, dw\right].$$

We make the change of variable $z = n^{-1/4}(w + A_n B_n^{-1})$ in the integral and we get

$$E_n = \frac{n^{1/4}}{Z_n\sqrt{2\pi}}\,\mathbb{E}\left[\mathbb{1}_{\{B_n^2>0\}} \int_{\mathbb{R}} f(z) \exp\left(-\frac{\sqrt{n}z^2}{2} + \frac{zn^{1/4}A_n}{B_n}\right) dz\right].$$

Let $U_1, \dots, U_n, \varepsilon_1, \dots, \varepsilon_n$ be independent random variables such that the distribution of $U_i$ is $\rho$ and the distribution of $\varepsilon_i$ is $(\delta_{-1} + \delta_1)/2$, for any $i \in \{1, \dots, n\}$. Since $\rho$ is symmetric, the random variables $\varepsilon_1 U_1, \dots, \varepsilon_n U_n$ are also independent with common distribution $\rho$. As a consequence

$$E_n = \frac{n^{1/4}}{Z_n\sqrt{2\pi}}\,\mathbb{E}\left[\mathbb{1}_{\{B_n^2>0\}} \int_{\mathbb{R}} f(z) \exp\left(-\frac{\sqrt{n}z^2}{2} + \sum_{i=1}^n \frac{zn^{1/4}\varepsilon_i U_i}{B_n}\right) dz\right].$$

For any $i \in \{1, \dots, n\}$, we denote (in the case where $B_n^2 = U_1^2 + \cdots + U_n^2 > 0$)

$$\forall z \in \mathbb{R} \qquad z_{i,n} = \frac{z n^{1/4} U_i}{\sqrt{U_1^2 + \cdots + U_n^2}}.$$

By using Fubini's theorem and the independence of $\varepsilon_i, U_i$, $i \in \{1, \dots, n\}$, we obtain

$$\begin{aligned} E_n = \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E}\Bigg[ \mathbb{1}_{\{B_n^2 > 0\}} \int_{\mathbb{R}} f(z) \exp\left(-\frac{\sqrt{n} z^2}{2}\right) \\ \times \mathbb{E}\left( \prod_{i=1}^n \exp\left(z_{i,n}\varepsilon_i\right) \,\middle|\, (U_1, \dots, U_n) \right) dz \Bigg]. \\ = \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E}\left[ \mathbb{1}_{\{B_n^2 > 0\}} \int_{\mathbb{R}} f\left(z\right) \exp\left(-\frac{\sqrt{n} z^2}{2}\right) \exp\left(\sum_{i=1}^n \ln\cosh\left(z_{i,n}\right)\right) dz \right]. \end{aligned}$$

We define the function $g$ by

$$\forall y \in \mathbb{R} \qquad g(y) = \ln\cosh y - \frac{y^2}{2}.$$

It is easy to see that $g(y) < 0$ for $y > 0$. We notice that $z_{1,n}^2 + \cdots + z_{n,n}^2 = \sqrt{n} z^2$ for any $z \in \mathbb{R}$, so that

$$E_n = \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E}\left[ \mathbb{1}_{\{B_n^2 > 0\}} \int_{\mathbb{R}} f\left(z\right) \exp\left(\sum_{i=1}^n g(z_{i,n})\right) dz \right].$$

Now we use Laplace's method. Let us examine the convergence of the term in the exponential: for any $i \in \{1, \dots, n\}$ and $z \in \mathbb{R}$, the Taylor-Lagrange formula states that there exists a random variable $\xi_i$ such that

$$g(z_{i,n}) = -\frac{z_{i,n}^4}{12} + \frac{z_{i,n}^5}{5!} g^{(5)}(\xi_i).$$

By a simple computation, we see that the function $g^{(5)}$ is bounded over $\mathbb{R}$. As a consequence

$$\sum_{i=1}^n g(z_{i,n}) = -\frac{z^4}{12} \frac{(Y_1^4 + \cdots + Y_n^4)/n}{((Y_1^2 + \cdots + Y_n^2)/n)^2} + \frac{z^5 \, (Y_1^5 + \cdots + Y_n^5)/n}{((Y_1^2 + \cdots + Y_n^2)/n)^{5/2}} O\left(\frac{1}{n^{1/4}}\right).$$

By hypothesis, the distribution $\rho$ has a finite fifth moment. Hence the law of large numbers implies that

$$\forall z \in \mathbb{R} \qquad \sum_{i=1}^n g(z_{i,n}) \underset{n \to +\infty}{\longrightarrow} -\frac{\mu_4 z^4}{12\sigma^4} \qquad \text{a.s.}$$

**Lemma 3.** *There exists $c > 0$ such that*

$$\forall z \in \mathbb{R} \quad \forall n \geq 1 \qquad \sum_{i=1}^n g(z_{i,n}) \leq -\frac{c z^4}{1 + z^2/\sqrt{n}}.$$

**Proof.** We define $h$ by

$$\forall y \in \mathbb{R}\backslash\{0\} \qquad h(y) = \frac{1+y^2}{y^4} g(y).$$

It is a negative continuous function on $\mathbb{R}\backslash\{0\}$. Since $g(y) \sim -y^4/12$ in the neighbourhood of 0, the function $h$ can be extended to a function continuous on $\mathbb{R}$ by putting $h(0) = -1/12$. Next we have

$$\forall y \in \mathbb{R}\backslash\{0\} \qquad h(y) = \frac{1+y^2}{y^2} \times \left( \frac{\ln \cosh y}{y^2} - \frac{1}{2} \right),$$

so that $h(y)$ goes to $-1/2$ when $|y|$ goes to $+\infty$. Therefore $h$ is bounded by some constant $-c$ with $c > 0$. Next we easily check that $x \longmapsto x^2/(1+x)$ is convex on $[0, +\infty[$ so that, for any $z \in \mathbb{R}$ and $n \geq 1$,

$$\sum_{i=1}^{n} g(z_{i,n}) \leq -nc \frac{1}{n} \sum_{i=1}^{n} \frac{z_{i,n}^4}{1+z_{i,n}^2} \leq -nc \frac{\left(\frac{1}{n}\sum_{i=1}^{n} z_{i,n}^2\right)^2}{1+\frac{1}{n}\sum_{i=1}^{n} z_{i,n}^2} = -\frac{cz^4}{1+z^2/\sqrt{n}},$$

since $z_{1,n}^2 + \cdots + z_{n,n}^2 = \sqrt{n} z^2$. □

If $|z| \leq n^{1/4}$ then $1 + z^2/\sqrt{n} \leq 2$ and thus, by the previous lemma,

$$\left| \mathbb{1}_{\{B_n^2>0\}} \mathbb{1}_{|z|\leq n^{1/4}} \exp\left( \sum_{i=1}^{n} g(z_{i,n}) \right) \right| \leq \exp\left( -\frac{cz^4}{2} \right).$$

Since

$$\mathbb{E}\left[ \int_{\mathbb{R}} \|f\|_\infty \exp\left( -\frac{cz^4}{2} \right) \, dz \right] < +\infty,$$

the dominated convergence theorem implies that

$$\mathbb{E}\left[ \mathbb{1}_{\{B_n^2>0\}} \int_{\mathbb{R}} \mathbb{1}_{|z|\leq n^{1/4}} f(z) \exp\left( \sum_{i=1}^{n} g(z_{i,n}) \right) \, dz \right] \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}} f(z) \exp\left( -\frac{\mu_4 z^4}{12\sigma^4} \right) \, dz.$$

If $|z| > n^{1/4}$ then $1 + z^2/\sqrt{n} \leq 2z^2/\sqrt{n}$ and thus, by the previous lemma,

$$\left| \mathbb{1}_{\{B_n^2>0\}} \mathbb{1}_{|z|>n^{1/4}} \exp\left( \sum_{i=1}^{n} g(z_{i,n}) \right) \right| \leq \exp\left( -\frac{c\sqrt{n}z^2}{2} \right).$$

Hence

$$\mathbb{E}\left[ \mathbb{1}_{\{B_n^2>0\}} \int_{\mathbb{R}} \mathbb{1}_{|z|>n^{1/4}} f(z) \exp\left( \sum_{i=1}^{n} g(z_{i,n}) \right) \, dz \right] \leq \frac{\|f\|_\infty \sqrt{2\pi}}{n^{1/4}\sqrt{c}},$$

and thus

$$\mathbb{E}\left[ \mathbb{1}_{\{B_n^2>0\}} \int_{\mathbb{R}} f(z) \exp\left( \sum_{i=1}^{n} g(z_{i,n}) \right) \, dz \right] \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}} f(z) \exp\left( -\frac{\mu_4 z^2}{12\sigma^4} \right) \, dz.$$

If we take $f = 1$, we get

$$\frac{Z_n\sqrt{2\pi}}{n^{1/4}} \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}} \exp\left(-\frac{\mu_4 z^4}{12\sigma^4}\right)\,dz.$$

We have proved that

$$\frac{W}{n^{1/4}} + \frac{S_n}{n^{1/4}\sqrt{T_n}} \overset{\mathscr{L}}{\underset{n\to\infty}{\longrightarrow}} \left(\int_{\mathbb{R}} \exp\left(-\frac{\mu_4 z^4}{12\sigma^4}\right)\,dz\right)^{-1} \exp\left(-\frac{\mu_4}{12\sigma^4}s^4\right)\,ds.$$

Since $(n^{-1/4}W)_{n\geq 1}$ converges in distribution to 0, Slutsky lemma (theorem 3.9 of [1]) implies that

$$\frac{S_n}{n^{1/4}\sqrt{T_n}} \overset{\mathscr{L}}{\underset{n\to\infty}{\longrightarrow}} \left(\int_{\mathbb{R}} \exp\left(-\frac{\mu_4 z^4}{12\sigma^4}\right)\,dz\right)^{-1} \exp\left(-\frac{\mu_4}{12\sigma^4}s^4\right)\,ds.$$

By an ultimate change of variables we compute that

$$\int_{\mathbb{R}} \exp\left(-\frac{\mu_4 z^4}{12\sigma^4}\right)\,dz = \left(\frac{3\sigma^4}{4\mu_4}\right)^{1/4} \Gamma\left(\frac{1}{4}\right).$$

This ends the proof of theorem 1.

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